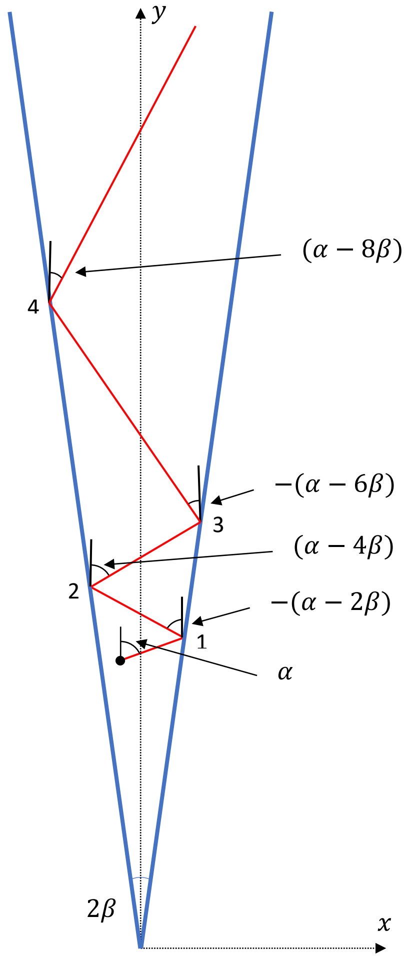
## Reflections in Parallel Vertical Mirrors Placed at an Angle




The aim of this note is to provide the coordinates of the points at which a ray from an object to an observer is reflected by the two mirrors. The two mirrors are each at an angle of $\beta$ to the vertical as shown and the ray from the viewed object travels at an angle $\alpha$ measured clockwise from the $y$ axis (all angles for paths along the ray will be measured in the same way).

If we number the points at which the ray strikes the mirror from 1 upwards, we can see that odd numbered points lie on the right mirror and even ones on the left. By tracking the triangles, it is fairly easy to show that:

$$\alpha_k = \begin{cases} -(\alpha - 2k\beta) & k \text{ odd} \\ (\alpha - 2k\beta) & k \text{ even} \end{cases}$$

For the object at point $(x_0, y_0)$, consider the distance $(l_1)$ of the first point on the right mirror from the vertex measured along the mirror:

$$l_1 \cos\beta = y_0 + r\cos\alpha$$

$$l_1 \sin\beta = x_0 + r\sin\alpha$$

where $r$ is the length of the ray from the object to the point. Eliminating $r$ now gives:

$$l_1 = \frac{y_0 \sin\alpha - x_0 \cos\alpha}{\sin(\alpha - \beta)} = \frac{s}{\sin(\alpha - \beta)}$$

where we define $s$ as $y_0 \sin\alpha - x_0 \cos\alpha$. Now consider the length $(r)$ of the ray between points 1 and 2:

$$l_2 \cos\beta = l_1 \cos\beta + r\cos(\alpha - 2\beta)$$

$$-l_2 \sin\beta = l_1 \sin\beta - r\sin(\alpha - 2\beta)$$

Eliminating $r$ now gives:

$$l_2 = l_1 \frac{\sin(\alpha - \beta)}{\sin(\alpha - 3\beta)} = \frac{s}{\sin(\alpha - 3\beta)}$$

Continuing in this way we can show that:

$$l_k = \frac{s}{\sin\{\alpha - (2k-1)\beta\}}$$

with the angle of the ray from this point on being given above $(\alpha_k)$. The ray will travel to infinity when a segment on its path has an angle $(\gamma)$ that puts it between the mirrors: $-\beta < \gamma < \beta$. This will occur for a $k$ value at which the formula for $l_k$ gives a negative value. If this final ray passes through the point $(x_f, y_f)$ we have:

$$(-)^{k-1} x_f = l_k \sin\beta - r\sin(\alpha - 2k\beta)$$

$$y_f = l_k \cos\beta + r\cos(\alpha - 2k\beta)$$

where the sign of $x_f$ is positive when $k$ is odd (right mirror) and negative when $k$ is even (left mirror). Eliminating $r$ now gives:

$$(-)^{k-1} x_f \cos(\alpha - 2k\beta) + y_f \sin(\alpha - 2k\beta) = l_k \sin\{\alpha - (2k-1)\beta\} = s$$

If we express the object and observer positions in polar coordinates with $(x_0, y_0)$ as $(r_0, \epsilon_0)$ and $(x_f, y_f)$ as $(r_f, \epsilon_f)$, then this becomes:

$$r_f \sin\{\alpha - 2k\beta + (-)^{k-1}\epsilon_f\} = s = r_0 \sin(\alpha - \epsilon_0)$$

Solving for the initial ray angle ($\alpha$) at the object now gives:

$$\alpha = \tan^{-1}\left(\frac{r_f \sin\{2k\beta + (-)^k \epsilon_f\} - r_0 \sin\epsilon_0}{r_f \cos\{2k\beta + (-)^k \epsilon_f\} - r_0 \cos\epsilon_0}\right)$$

To find a ray path from the object to the observer, this equation has to be solved for the initial ray angle $\alpha$ and, since the $\pm$ sign is determined by the last mirror the ray hits, solutions have to be considered for both signs. For an observer at infinity this becomes:

$$a_k = 2k\beta + (-)^k \epsilon_f$$

This analysis is based on a first reflection off the right mirror. It can be used to analyse ray paths for rays that hit the left mirror first by inverting the $x$ coordinates of the initial and final points and then inverting all the $x$ values for the calculated points on ray paths.